

# Discrete uniformizing metrics on distributional limits of sphere packings

James R. Lee  
University of Washington

## Abstract

Suppose that  $\{G_n\}$  is a sequence of finite graphs such that each  $G_n$  is the tangency graph of a sphere packing in  $\mathbb{R}^d$ . Let  $\rho_n$  be a uniformly random vertex of  $G_n$  and suppose that  $(G, \rho)$  is the distributional limit of  $\{(G_n, \rho_n)\}$  in the sense of Benjamini and Schramm. Then the conformal growth exponent of  $(G, \rho)$  is at most  $d$ . In other words, there exists a unimodular “unit volume” weighting of the graph metric on  $(G, \rho)$  such that the volume growth of balls is asymptotically polynomial with exponent  $d$ . This generalizes to limits of graphs that can be “quasisymmetrically” packed in an Ahlfors  $d$ -regular metric measure space.

In particular, a result of the author implies that under moment conditions on the degree of the root  $\rho$ , the almost sure spectral dimension of  $G$  is at most  $d$ . As another application, we show that the spectrum of finite graphs sphere-packed in  $\mathbb{R}^d$  is dominated by a variant of the  $d$ -dimensional Weyl law.

## 1 Introduction

The theory of random planar graphs has been an active area of study in the last twenty years (see, e.g., [Ben10]), inspired partially by the connection to two-dimensional quantum gravity [ADJ97]. As noted by Benjamini and Curien [BC11], an analogous theory in higher dimensions has proved elusive, in part based on the difficulty of enumeration for higher-dimensional simplicial complexes (see [BZ11] and the references therein).

To address this discrepancy, the authors of [BC11] explored the extension of analytic and probabilistic methods based on potential theory. A graph  $G$  is said to be *sphere-packed in  $\mathbb{R}^d$*  if  $G$  is the tangency graph of a collection of interior-disjoint spheres in  $\mathbb{R}^d$ . Benjamini and Curien proved that if a family of finite graphs can be sphere-packed in  $\mathbb{R}^d$  with spheres of bounded aspect ratio (so that the ratio of the radii of tangent spheres is  $O(1)$ ), then a distributional limit of such graphs is *d-parabolic*.

Roughly speaking, *d-parabolicity* means that the  $L^d$  extremal length from a fixed vertex to  $\infty$  is infinite, where the  $L^d$  extremal length is a natural analog Cannon’s vertex extremal length [Can94] (the case  $d = 2$ ); see also [Duf62] and Section 1.3. It is well-known that the special case of 2-parabolicity carries strong probabilistic significance; for instance, for graphs with uniformly bounded degrees, 2-parabolicity is equivalent to recurrence of the random walk (see [Duf62, DS84]). Unfortunately, for  $d > 2$ , the theory of  $L^d$  extremal length is somewhat more unwieldy, and is not known to yield such control on the random walk.

In this work, we study a related notion that one might refer to as the “extremal growth rate.” For graphs that can be sphere-packed in  $\mathbb{R}^d$ , we show that it is possible to construct metrics that

uniformize their underlying geometry so that the *counting measure* has  $d$ -dimensional volume growth. Employing the results of [Lee17a], one does obtain substantial probabilistic consequences, including  $d$ -dimensional lower bounds on the diagonal heat kernel (see Theorem 1.4 below). Moreover, our results hold in considerable generality; they require no assumption on the degree or aspect ratio of the packing, and they extend to graphs that can be “quasisymmetrically” packed in an Ahlfors  $d$ -regular metric measure space.

The method of proof is based partially on a celebrated lemma of Benjamini and Schramm [BS01]. They show that if  $\{G_n\}$  is a sequence of finite planar triangulations with uniformly bounded degrees and  $\{G_n\}$  converges to a distributional limit  $(G, \rho)$ , then almost surely any circle packing of  $G$  has at most one accumulation point in the plane. An analogous result holds for graphs sphere-packed in  $\mathbb{R}^d$  when  $d > 2$  [BC11].

We argue that, in a quantitative sense, as long as the accumulation points remain separated, one can construct a multi-scale reweighting of the spheres in the packing, endowing the graph with a metric that reflects its  $d$ -dimensional structure with respect to the underlying counting measure. This is carried out in Section 2.

In Section 3, we discuss briefly a conjecture that we call the “systolic minor problem” based on analogies with systolic geometry. This problem arises partly out of an attempt to connect the construction of Section 2 to the more topological methods of [BLR10, KLPT11, Mat15, Lee16] as explored in [Lee17a, Lee17b], and to extend the isoperimetric theorem of [BP11] to higher dimensions.

## 1.1 Discrete conformal metrics on sphere-packed graphs

Consider a locally finite, connected graph  $G$ . A *conformal metric on  $G$*  is a map  $\omega : V(G) \rightarrow \mathbb{R}_+$ . This endows  $G$  with a graph distance as follows: Give to every edge  $\{u, v\} \in E(G)$  a length  $\text{len}_\omega(\{u, v\}) := \frac{1}{2}(\omega(u) + \omega(v))$ . This prescribes to every path  $\gamma = \{v_0, v_1, v_2, \dots\}$  in  $G$  the induced length

$$\text{len}_\omega(\gamma) = \sum_{k \geq 0} \text{len}_\omega(\{v_k, v_{k+1}\}).$$

Now for  $u, v \in V(G)$ , one defines the path metric  $\text{dist}_\omega(u, v)$  as the infimum of the lengths of all  $u$ - $v$  paths in  $G$ . Denote the closed ball

$$B_\omega(x, R) = \{y \in V(G) : \text{dist}_\omega(x, y) \leq R\}.$$

We can now state a special case of our main technical theorem; the connection to distributional limits and random walks is discussed subsequently.

**Theorem 1.1.** *For every  $d \geq 1$ , there is a constant  $C = C(d)$  such that the following holds. If  $G = (V, E)$  is a finite graph that can be sphere-packed in  $\mathbb{R}^d$ , then there is a conformal metric  $\omega : V \rightarrow \mathbb{R}_+$  that satisfies*

$$\frac{1}{|V|} \sum_{x \in V} \omega(x)^d = 1,$$

and such that

$$\max_{x \in V(G)} |B_\omega(x, R)| \leq CR^d (\log R)^2 \quad \forall R \geq 1.$$

## 1.2 Conformal growth exponents

If  $(G, \rho)$  is random rooted graph, then a *conformal metric* on  $(G, \rho)$  is a random triple  $(G', \omega, \rho')$  with  $\omega : V(G) \rightarrow \mathbb{R}_+$  such that  $(G, \rho)$  and  $(G', \rho')$  have the same law. We say that the conformal weight is *normalized* if  $\mathbb{E}[\omega(\rho)^2] = 1$ . One thinks of such a metric  $\omega : V(G) \rightarrow \mathbb{R}_+$  as deforming the geometry of the underlying graph subject to a bound on the total “area.” As shown in [Lee17a], normalized conformal metrics with nice geometric properties form a powerful tool in understanding the structure of  $(G, \rho)$ .

In the present work, we consider *unimodular* random graphs (see Section 1.6); such graphs arise as distributional limits of finite random rooted graphs  $\{(G_n, \rho_n)\}$  where  $\rho_n \in V(G_n)$  is chosen uniformly at random. We will consider only unimodular conformal metrics  $\omega$  on  $(G, \rho)$ ; in other words, the setting where  $(G, \omega, \rho)$  is unimodular as a marked graph.

**Conformal growth exponents.** Consider a unimodular random graph  $(G, \rho)$ . In [Lee17a], we defined the *upper and lower conformal growth exponents* of  $(G, \rho)$ , respectively, by

$$\overline{\dim}_{\text{cg}}(G, \rho) := \inf_{\omega} \limsup_{R \rightarrow \infty} \frac{\log \|\#B_{\omega}(\rho, R)\|_{L^\infty}}{\log R}, \quad (1.1)$$

$$\underline{\dim}_{\text{cg}}(G, \rho) := \inf_{\omega} \liminf_{R \rightarrow \infty} \frac{\log \|\#B_{\omega}(\rho, R)\|_{L^\infty}}{\log R}, \quad (1.2)$$

where the infimum is over all normalized unimodular conformal metrics on  $(G, \rho)$ , and we use  $\|X\|_{L^\infty}$  to denote the essential supremum of a random variable  $X$ , and  $\#S$  to denote the cardinality of a set  $S$ .

When  $\overline{\dim}_{\text{cg}}(G, \rho) = \underline{\dim}_{\text{cg}}(G, \rho)$ , define the *conformal growth exponent* by

$$\dim_{\text{cg}}(G, \rho) := \overline{\dim}_{\text{cg}}(G, \rho) = \underline{\dim}_{\text{cg}}(G, \rho).$$

Note that the quantities  $\overline{\dim}_{\text{cg}}$ ,  $\underline{\dim}_{\text{cg}}$ ,  $\dim_{\text{cg}}$  are functions of the law of  $(G, \rho)$ ; they are not defined on (fixed) rooted graphs.

Note that the conformal growth exponent bears a philosophical resemblance to Pansu’s notion of *conformal dimension* [Pan89]. We refer to Pansu’s recent work [Pan16] which explores in detail the relationship between sphere packings and the theory of large-scale conformal maps.

**$L^q$  conformal growth rate.** Let us define a generalization: If  $(G, \omega, \rho)$  is a unimodular random conformal graph, we denote

$$\|\omega\|_{L^q} := (\mathbb{E} \omega(\rho)^q)^{1/q}.$$

Say that  $\omega$  is  $L^q$ -normalized if  $\|\omega\|_{L^q} = 1$ .

Define the analogous  $L^q$  quantities:  $\overline{\dim}_{\text{cg}}^q, \underline{\dim}_{\text{cg}}^q, \dim_{\text{cg}}^q$  where now the infima in (1.1) and (1.2) are over all  $L^q$ -normalized conformal metrics on  $(G, \rho)$ . Observe that, by monotonicity of  $L^q$  norms, we have

$$q \leq q' \implies \dim_{\text{cg}}^q(G, \rho) \leq \dim_{\text{cg}}^{q'}(G, \rho).$$

The next theorem constitutes the main new result presented here. We use  $\Rightarrow$  to denote convergence in the distributional sense; see Section 1.6.

**Theorem 1.2.** *For any  $d \geq 1$ , the following holds. If  $\{G_n\}$  are finite graphs that can be sphere-packed in  $\mathbb{R}^d$  and  $\{G_n\} \Rightarrow (G, \rho)$ , then there is an  $L^d$ -normalized unimodular conformal metric  $\omega : V(G) \rightarrow \mathbb{R}_+$  such that for all  $R \geq 1$ ,*

$$\|\#B_{\omega}(\rho, R)\|_{L^\infty} \leq O(R^d (\log R)^2). \quad (1.3)$$

*In particular,  $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$ .*

For  $d = 1$ , the theorem is uninteresting; the trivial weight  $\omega \equiv 1$  suffices. For  $d \geq 2$ , the last assertion follows from  $\overline{\dim}_{\text{cg}}(G, \rho) = \overline{\dim}_{\text{cg}}^2(G, \rho) \leq \overline{\dim}_{\text{cg}}^d(G, \rho)$ .

We remark that some  $(\log R)^{O(1)}$  factor is necessary even for the case  $d = 2$  (planar graphs); see [Lee17a, §2]. In fact, we prove Theorem 1.2 in somewhat greater generality: For graphs that are “quasisymmetrically” packed in an Ahlfors  $d$ -regular metric measure space using bodies that are appropriately “round” (see Section 2 for the precise definitions).

A primary motivation for Theorem 1.2 is that such metrics can be used to obtain estimates on the almost sure spectral dimension of  $G$ . For a locally finite, connected graph  $G$ , denote the discrete-time heat kernel

$$p_T^G(x, y) := \mathbb{P}[X_T = y \mid X_0 = x],$$

where  $\{X_n\}$  is the standard random walk on  $G$ . We recall the *spectral dimension* of  $G$ :

$$\dim_{\text{sp}}(G) := \lim_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n},$$

whenever the limit exists. If the limit does exist, then it is the same for all  $x \in V(G)$ .

Say that a real-valued random variable  $X$  has *negligible tails* if its tails decay faster than any inverse polynomial:

$$\lim_{n \rightarrow \infty} \frac{\log n}{|\log \mathbb{P}[|X| > n]|} = 0, \quad (1.4)$$

where we take  $\log(0) = -\infty$  in the preceding definition (in the case that  $X$  is essentially bounded). The next theorem is from [Lee17a]; it asserts that if  $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$ , then almost surely  $G$  admits  $d$ -dimensional lower bounds on the diagonal heat kernel:

$$p_{2n}^G(\rho, \rho) \geq n^{-d/2+o(1)} \quad \text{as } n \rightarrow \infty.$$

**Theorem 1.3.** *Suppose that  $(G, \rho)$  is a unimodular random graph such that  $\deg_G(\rho)$  has negligible tails. Then almost surely:*

$$\limsup_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n} \leq \overline{\dim}_{\text{cg}}(G, \rho).$$

*In particular, if there is a number  $d$  such that almost surely  $\dim_{\text{sp}}(G) = d$ , then  $d \leq \overline{\dim}_{\text{cg}}(G, \rho)$ .*

In certain situations, one can give stronger estimates. Indeed, when the conformal growth rate has only a polylogarithmic correction as in (1.3), one obtains stronger results (see [Lee17a, §4.2]).

**Theorem 1.4.** *Suppose  $(G, \rho)$  is the distributional limit of finite graphs that can be sphere-packed in  $\mathbb{R}^d$ , and that  $\deg_G(\rho)$  has exponential tails. Then there is a constant  $c \geq 1$  such that for  $n$  sufficiently large,*

$$\mathbb{P} \left[ p_{2n}^G(\rho, \rho) \geq \frac{n^{-d/2}}{(\log n)^c} \right] \geq 1 - \frac{1}{\log n}.$$

### 1.3 Gauged conformal growth and $d$ -parabolicity

Consider a locally finite connected graph  $G = (V, E)$ . Let  $\Gamma_G$  denote a collection of simple paths in  $G$ . The  $\ell_d$ -vertex extremal length of  $\Gamma_G$  is defined by

$$\text{VEL}_d(\Gamma_G) := \sup_{\omega} \inf_{\gamma \in \Gamma_G} \frac{\text{len}_{\omega}(\gamma)}{\|\omega\|_{\ell_d(V)}},$$

where the infimum is over all conformal metrics on  $G$ , and  $\|\omega\|_{\ell_d(V)} = \left(\sum_{v \in V} \omega(v)^d\right)^{1/d}$ .

Fix a vertex  $v_0 \in V$  and let  $\Gamma_G(v_0)$  denote the set of infinite simple paths in  $G$  emanating from  $v_0$ . One says that  $G$  is  $d$ -parabolic if  $\text{VEL}_d(\Gamma_G(v_0)) = \infty$  (see [HS95, BS13]). One can check that this definition does not depend on the choice of  $v_0 \in V$ .

Consider a sequence  $\{G_n\}$  of finite graphs with uniformly bounded degrees. Furthermore, suppose that each  $G_n$  is sphere-packed in  $\mathbb{R}^d$  and  $\{G_n\} \Rightarrow (G, \rho)$ . There are examples where  $G$  is almost surely 2-parabolic, but  $\underline{\dim}_{\text{cg}}^d(G, \rho) \geq \underline{\dim}_{\text{cg}}(G, \rho) = \infty$ , and other examples where  $\dim_{\text{cg}}(G, \rho) = d \geq 2$  but  $G$  is almost surely not  $d$ -parabolic; see Section 2.2.

**Gauged growth.** On the other hand, there is a common strengthening of the conditions. Say that  $(G, \rho)$  has  $(C, R, d)$ -growth if there is an  $L^d$ -normalized conformal metric  $\omega : V(G) \rightarrow \mathbb{R}_+$  such that

$$\|\#B_\omega(\rho, R)\|_{L^\infty} \leq CR^d. \quad (1.5)$$

Say that  $(G, \rho)$  has *gauged  $d$ -dimensional conformal growth* if there is a constant  $C \geq 1$  such that  $(G, \rho)$  has  $(C, R, d)$ -growth for all  $R \geq 0$ . A sequence  $\{(G_n, \rho_n)\}$  has *uniform gauged  $d$ -dimensional conformal growth* if there is a constant  $C \geq 1$  such that  $(G_n, \rho_n)$  has  $(C, R, d)$ -growth for all  $R \geq 0$  and  $n \geq 1$ .

It is straightforward to see that if  $(G, \rho)$  has gauged  $d$ -dimensional growth, then  $\overline{\dim}_{\text{cg}}^d(G, \rho) \leq d$ . For each  $k \geq 0$ , let  $\omega_k$  denote an  $L^d$ -normalized conformal metric on  $(G, \rho)$  satisfying (1.5) and define

$$\hat{\omega} := \left( \frac{6}{\pi^2} \sum_{k \geq 0} \frac{\omega_k^d}{k^2} \right)^{1/d}.$$

Establishing  $d$ -parabolicity is somewhat more involved; the  $d = 2$  case of the following theorem is [Lee17a, Thm. 2.1].

**Theorem 1.5.** *For every  $d \geq 2$ , the following holds. If  $(G, \rho)$  is a unimodular random graph such that  $\deg_G(\rho)$  is essentially bounded and  $(G, \rho)$  has gauged  $d$ -dimensional conformal growth, then  $G$  is almost surely  $d$ -parabolic.*

In order to establish Theorem 1.2, we prove the following stronger statement; see Corollary 2.2.

**Theorem 1.6.** *Suppose that  $\{G_n\}$  is a sequence of finite graphs and that each  $G_n$  admits a sphere-packing in  $\mathbb{R}^d$ . If  $\{G_n\} \Rightarrow (G, \rho)$ , then  $(G, \rho)$  has gauged  $d$ -dimensional conformal growth.*

## 1.4 The Laplacian spectrum of finite graphs

Let  $G = (V, E)$  denote a finite connected graph and let  $n = |V|$ . Let  $\{1 - \lambda_k(G) : k = 0, 1, \dots, n-1\}$  be the eigenvalues of the random walk operator on  $G$ , where

$$0 = \lambda_0(G) \leq \lambda_1(G) \leq \dots \leq \lambda_{n-1}(G).$$

Define also

$$\Delta_G(k) := \max_{S \subseteq V: |S| \leq k} \sum_{x \in S} \deg_G(x),$$

where  $\deg_G(x)$  denotes the degree of a vertex  $x \in V$ .

In [KLPT11], it is shown that there is a constant  $C > 0$  such that if  $G$  is a planar graph, then for all  $k = 1, \dots, n-1$ ,

$$\lambda_k \leq C \Delta_G(1) \frac{k}{n},$$

where  $\Delta_G(1)$  is the maximum degree in  $G$ . The estimate for  $k = 2$  is due to Spielman and Teng [ST07]. In [Lee17a], the author improves this bound to

$$\lambda_k \leq C \frac{\Delta_G(k)}{n}.$$

While the utility of this improvement is not immediately apparent, the correct quantitative dependence is essential to a spectral argument proving that the uniform infinite planar triangulation is almost surely recurrent [Lee17a]; this fact was first established by Gurel-Gurevich and Nachmias [GN13] using effective resistance estimates.

In Section 2.3, we use Theorem 1.1 to prove an analogous statement for graphs sphere-packed in  $\mathbb{R}^d$ .

**Theorem 1.7.** *For every  $d \geq 2$ , there is a constant  $c_d$  such that the following holds. If  $G$  is an  $n$ -vertex graph that can be sphere-packed in  $\mathbb{R}^d$ , then for  $k = 1, 2, \dots, n-1$ ,*

$$\lambda_k(G) \leq c_d \frac{\Delta_G(k)}{k} \left( \log \frac{n}{k} \right)^2 \left( \frac{k}{n} \right)^{2/d}.$$

Note that  $\Delta_G(k)/k$  is the average degree of the  $k$  vertices of largest degree in  $G$ . Up to the factor of  $(\log(n/k))^2$ , this bound is tight for a  $d$ -dimensional box  $\{1, 2, \dots, n^{1/d}\}^d$  considered as a subgraph of the integer lattice  $\mathbb{Z}^d$ . Whether the  $(\log(n/k))^2$  factor can be removed from the bound is an interesting open question.

## 1.5 Preliminaries

We use the notations  $\mathbb{R}_+ = [0, \infty)$  and  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ .

All graphs appearing in this paper are undirected and locally finite and without loops or multiple edges. If  $G$  is such a graph, we use  $V(G)$  and  $E(G)$  to denote the vertex and edge set of  $G$ , respectively. If  $S \subseteq V(G)$ , we use  $G[S]$  for the induced subgraph on  $S$ . For  $A, B \subseteq V(G)$ , we write  $E_G(A, B)$  for the set of edges with one endpoint in  $A$  and the other in  $B$ . We write  $\text{dist}_G$  for the unweighted path metric on  $V(G)$ , and  $B_G(x, r) = \{y \in V(G) : \text{dist}_G(x, y) \leq r\}$  to denote the closed  $r$ -ball around  $x \in V(G)$ . Also let  $\deg_G(x)$  denote the degree of a vertex  $x \in V(G)$ , and  $d_{\max}(G) = \sup_{x \in V(G)} \deg_G(x)$ .

Write  $G_1 \cong G_2$  to denote that  $G_1$  and  $G_2$  are isomorphic as graphs. If  $(G_1, \rho_1)$  and  $(G_2, \rho_2)$  are rooted graphs, we write  $(G_1, \rho_1) \cong_\rho (G_2, \rho_2)$  to denote the existence of a rooted isomorphism.

## 1.6 Unimodular random graphs and distributional limits

We begin with a discussion of unimodular random graphs and distributional limits. One may consult the extensive reference of Aldous and Lyons [AL07]. The paper [BS01] offers a concise introduction to distributional limits of finite planar graphs. We briefly review some relevant points.

Let  $\mathcal{G}$  denote the set of isomorphism classes of connected, locally finite graphs; let  $\mathcal{G}_\bullet$  denote the set of *rooted* isomorphism classes of *rooted*, connected, locally finite graphs. Define a metric on  $\mathcal{G}_\bullet$  as follows:  $\text{loc}((G_1, \rho_1), (G_2, \rho_2)) = 1/(1 + \alpha)$ , where

$$\alpha = \sup \left\{ r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r) \right\}.$$

$(\mathcal{G}_\bullet, \text{loc})$  is a separable, complete metric space. For probability measures  $\{\mu_n\}, \mu$  on  $\mathcal{G}_\bullet$ , write  $\{\mu_n\} \Rightarrow \mu$  when  $\mu_n$  converges weakly to  $\mu$  with respect to  $\text{loc}$ .



**The Mass-Transport Principle.** Let  $\mathcal{G}_{\bullet\bullet}$  denote the set of doubly-rooted isomorphism classes of doubly-rooted, connected, locally finite graphs. A probability measure  $\mu$  on  $\mathcal{G}_{\bullet}$  is *unimodular* if it obeys the following *Mass-Transport Principle*: For all Borel-measurable  $F : \mathcal{G}_{\bullet\bullet} \rightarrow [0, \infty]$ ,

$$\int \sum_{x \in V(G)} F(G, \rho, x) d\mu((G, \rho)) = \int \sum_{x \in V(G)} F(G, x, \rho) d\mu((G, \rho)). \quad (1.6)$$

If  $(G, \rho)$  is a random rooted graph with law  $\mu$ , and  $\mu$  is unimodular, we say that  $(G, \rho)$  is a *unimodular random graph*; we will often omit the word “random” and simply refer to  $(G, \rho)$  as a unimodular random graph.

**Distributional limits of finite graphs.** As observed by Benjamini and Schramm [BS01], unimodular random graphs can be obtained from limits of finite graphs. Consider a sequence  $\{G_n\} \subseteq \mathcal{G}$  of finite graphs, and let  $\rho_n$  denote a uniformly random element of  $V(G_n)$ . Then  $\{(G_n, \rho_n)\}$  is a sequence of  $\mathcal{G}_{\bullet}$ -valued random variables, and one has the following.

**Lemma 1.8.** *If  $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ , then  $(G, \rho)$  is unimodular.*

If  $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ , we say that  $(G, \rho)$  is the *distributional limit* of the sequence  $\{(G_n, \rho_n)\}$ . When  $\{G_n\}$  is a sequence of finite graphs, we write  $\{G_n\} \Rightarrow (G, \rho)$  for  $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$  where  $\rho_n \in V(G_n)$  is chosen uniformly at random.

**Unimodular random conformal graphs.** A *conformal graph* is a pair  $(G, \omega)$ , where  $G$  is a connected, locally finite graph and  $\omega : V(G) \rightarrow \mathbb{R}_+$ . Let  $\mathcal{G}^*$  and  $\mathcal{G}_{\bullet}^*$  denote the collections of isomorphism classes of conformal graphs and conformal rooted graphs, respectively. As in Section 1.6, one can define a metric on  $\mathcal{G}_{\bullet}^*$  as follows:  ${}_{\text{loc}}^*((G_1, \omega_1, \rho_1), (G_2, \omega_2, \rho_2)) = 1/(\alpha + 1)$ , where

$$\alpha = \sup \left\{ r > 0 : B_{G_1}(\rho_1, r) \cong_{\rho} B_{G_2}(\rho_2, r) \text{ and } (\omega_1|_{B_{G_1}(\rho_1, r)}, \omega_2|_{B_{G_2}(\rho_2, r)}) \leq \frac{1}{r} \right\},$$

where for two weights  $\omega_1 : V(H_1) \rightarrow \mathbb{R}_+$  and  $\omega_2 : V(H_2) \rightarrow \mathbb{R}_+$  on rooted-isomorphic graphs  $(H_1, \rho_1)$  and  $(H_2, \rho_2)$ , we write

$$(\omega_1, \omega_2) := \inf_{\psi: V(H_1) \rightarrow V(H_2)} \|\omega_2 \circ \psi - \omega_1\|_{\ell^\infty},$$

and the infimum is over all graph isomorphisms from  $H_1$  to  $H_2$  satisfying  $\psi(\rho_1) = \rho_2$ .

If  $\{\mu_n\}$  and  $\mu$  are probability measures on  $\mathcal{G}_{\bullet}^*$ , we abuse notation and write  $\{\mu_n\} \Rightarrow \mu$  to denote weak convergence with respect to  ${}_{\text{loc}}^*$ . One defines unimodularity of a random rooted conformal graph  $(G, \omega, \rho)$  analogously to (1.6): It should now hold that for all Borel-measurable  $F : \mathcal{G}_{\bullet\bullet}^* \rightarrow [0, \infty]$ ,

$$\int \sum_{x \in V(G)} F(G, \omega, \rho, x) d\mu((G, \omega, \rho)) = \int \sum_{x \in V(G)} F(G, \omega, x, \rho) d\mu((G, \omega, \rho)).$$

Indeed, such decorated graphs are a special case of the marked networks considered in [AL07], and again it holds that every distributional limit of finite unimodular random conformal graphs is a unimodular random conformal graph.

Suppose that  $(G, \rho)$  is a unimodular random graph. A *conformal weight* on  $(G, \rho)$  is a unimodular conformal graph  $(G', \omega, \rho')$  such that  $(G, \rho)$  and  $(G', \rho')$  have the same law. We will speak simply of a “conformal metric  $\omega$  on  $(G, \rho)$ .” Only such unimodular metrics are considered in this work.

### 1.6.1 Conformal growth rates under distributional limits

In order to establish our main result, we need to pass from a sequence of conformal metrics on finite graphs to a conformal metric on the distributional limit.

**Theorem 1.9.** *Consider  $d, q \geq 1$ . Suppose  $\{(G_n, \rho_n)\}$  is a sequence of finite unimodular random graphs and  $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ . If there is a function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $h(R) \leq R^{o(1)}$  as  $R \rightarrow \infty$ , and a sequence of  $L^q$ -normalized unimodular random conformal graphs  $\{(G_n, \omega_n, \rho_n)\}$  satisfying*

$$\|B_{\omega_n}(\rho_n, R)\|_{L^\infty} \leq R^d h(R), \quad (1.7)$$

*then  $\overline{\dim}_{\text{cg}}^q(G, \rho) \leq d$ .*

*If the unimodular random graphs  $\{(G_n, \rho_n)\}$  have uniform gauged  $d$ -dimensional growth, then  $(G, \rho)$  has gauged  $d$ -dimensional growth.*

The preceding theorem is established by the next two lemmas.

**Lemma 1.10.** *Consider a sequence  $\{(G_n, \omega_n, \rho_n)\}$  of unimodular random conformal graphs satisfying the following conditions:*

1. *For every  $n \geq 1$ ,  $\mathbb{E}[\omega_n(\rho_n)] \leq 1$ .*
2. *There is a sequence  $\{b_k : k \in \mathbb{N}\}$  such that for every  $k \geq 1$ ,  $\limsup_{n \rightarrow \infty} \|B_{G_n}(\rho_n, k)\|_{L^\infty} \leq b_k$ .*

*Then  $\{(G_n, \omega_n, \rho_n)\}$  has a subsequential weak limit.*

*Proof.* Let us first pass to a subsequence  $\{(G_n, \omega_n, \rho_n)\}$  satisfying  $\|B_{G_n}(\rho_n, k)\|_{L^\infty} \leq b_k$  for  $k \leq n$ .

For  $k \geq 1$  and a conformal graph  $(G, \omega)$  and  $x, y \in V(G)$ , define the flow

$$F_k(G, \omega, x, y) = \omega(x) \mathbf{1}_{\{\text{dist}_G(x, y) \leq k\}}.$$

Employing the Mass-Transport Principle, for  $k \leq n$  we have

$$\begin{aligned} \mathbb{E} \left[ \sum_{x \in B_{G_n}(\rho_n, k)} \omega_n(x) \right] &= \mathbb{E} \left[ \sum_{x \in V(G_n)} F_k(G_n, \omega_n, x, \rho_n) \right] \\ &= \mathbb{E} \left[ \sum_{x \in V(G_n)} F_k(G_n, \omega_n, \rho_n, x) \right] \\ &\leq b_k \mathbb{E}[\omega_n(\rho_n)] = b_k. \end{aligned}$$

In particular, for any  $\lambda > 0$ ,

$$\mathbb{P} \left[ \max_{x \in B_{G_n}(\rho_n, k)} \omega(x) > \lambda k^2 b_k \right] \leq \frac{1}{\lambda k^2 b_k}.$$

Denote the event

$$\mathcal{E}_n(\lambda) := \left\{ (G_n, \omega_n, \rho_n) : \max_{x \in B_{G_n}(\rho_n, k)} \omega(x) \leq \lambda k^2 b_k \quad \forall k = 1, 2, \dots, n \right\}.$$

A union bound yields

$$\mathbb{P}[\mathcal{E}_n(\lambda)] \geq 1 - O(1/\lambda). \quad (1.8)$$



Let  $\mu_n(\lambda)$  denote the law of  $(G_n, \omega_n, \rho_n)$  conditioned on  $\mathcal{E}_n(\lambda)$ . Then  $\{\mu_n(\lambda)\}$  admits a subsequential weak limit because the space of bounded weights on balls of radius  $k$  is compact (recalling that such balls almost surely have size at most  $b_k$  by assumption). Therefore by diagonalization, the sequence  $\{\mu_n(n)\}$  admits a subsequential weak limit  $\mu$ . Let  $\{n_j\}$  be a subsequence for which  $\{\mu_{n_j}(2^{n_j})\} \Rightarrow \mu$ . Then  $(G_{n_j}, \omega_{n_j}, \rho_{n_j}) \Rightarrow \mu$  as well since  $\mathbb{P}[\mathcal{E}_{n_j}(n_j)] \rightarrow 1$  by (1.8).  $\square$

**Theorem 1.9** now follows from the following lemma by taking  $\{\omega_n\}$  to be the  $L^q$ -normalized conformal weights that verify (1.7) for  $\{(G_n, \rho_n)\}$ .

**Lemma 1.11.** *Suppose  $\{(G_n, \rho_n)\}$  is a sequence of finite unimodular random graphs such that  $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ . Then any sequence of unimodular random conformal graphs  $\{(G_n, \omega_n, \rho_n)\}$  satisfying  $\mathbb{E}[\omega_n(\rho)] \leq 1$ , has a subsequential weak limit in the metric  $^*_{\text{loc}}$ .*

*Proof.* Since there are only countably many rooted isomorphism classes of rooted balls of radius  $r$ , we have

$$\lim_{b \rightarrow \infty} \mathbb{P}[\#B_G(\rho, r) > b] = 0. \quad (1.9)$$

In particular, there exists a sequence  $\{b_{j,\ell}\}$  such that

$$\lim_{\ell \rightarrow \infty} \mathbb{P}[\#B_G(\rho, j) \leq b_{j,\ell} \quad \forall j \geq 1] = 1. \quad (1.10)$$

Let  $(G_n^\ell, \omega_n^\ell, \rho_n^\ell)$  have the law of  $(G_n, \omega_n, \rho_n)$  conditioned on the event  $\{\#B_{G_n}(\rho_n, j) \leq b_{j,\ell} \quad \forall j \geq 1\}$  (for  $\ell$  sufficiently large that this event has non-zero probability). From (1.10), it suffices to prove that  $\{(G_n^\ell, \omega_n^\ell, \rho_n^\ell)\}$  has a weak subsequential limit for all  $\ell$  sufficiently large, and this is precisely the content of Lemma 1.10.  $\square$

## 1.7 Systems of dyadic cubes

Consider a pseudo-metric space  $(X, d)$  (i.e., we allow for the possibility that  $d(x, y) = 0$  when  $x \neq y$ ). Throughout the paper, we will deal only with complete, separable, pseudo-metric spaces. For  $x \in X$  and two subsets  $S, T \subseteq X$ , we use the notations  $d(S, T) = \inf_{x \in S, y \in T} d(x, y)$  and  $d(x, S) = d(\{x\}, S)$ . Define  $\text{diam}(S, d) = \sup_{x, y \in S} d(x, y)$  and for  $R \geq 0$ , define the closed balls

$$B_{(X,d)}(x, R) = \{y \in X : d(x, y) \leq R\}.$$

We omit the subscript  $(X, d)$  if the underlying metric space is clear from context. We say that  $(X, d)$  is *doubling* if there is a constant  $\mathcal{D}$  such that every closed ball in  $X$  can be covered by  $\mathcal{D}$  closed balls of half the radius, and we let  $\mathcal{D}_{(X,d)}$  denote the infimal  $\mathcal{D}$  for which this holds.

If  $\mu$  is a measure on the Borel  $\sigma$ -algebra of  $X$ , we refer to  $(X, d, \mu)$  as a *metric measure space*. Such a space is said to be *Ahlfors  $\beta$ -regular* if there are constants  $c_1, c_2 > 0$  such that

$$c_1 R^\beta \leq \mu(B(x, R)) \leq c_2 R^\beta \quad \forall x \in X, R \in [0, \text{diam}(X)].$$

It will occasionally be convenient to record the constants  $c_1, c_2$ , in which case we say that  $(X, d, \mu)$  is  $(c_1, c_2, \beta)$ -regular. We recall the following elementary fact:

**Fact 1.12.** *If  $(X, d, \mu)$  is Ahlfors  $\beta$ -regular for some  $\beta > 0$ , then  $(X, d)$  is doubling, and  $\mathcal{D}_{(X,d)} \leq C$  for some constant  $C = C(c_1, c_2, \beta)$  depending only on  $c_1, c_2, \beta$ .*

We will employ an appropriate system of hierarchical dyadic partitions of a doubling metric space  $(X, d)$  along the lines of [Chr90] and [Dav91]. Deterministic and randomized constructions of this type are a basic tool in harmonic analysis and metric geometry (see, e.g., [LN05] and [HK12]).

For our purposes, it will be easiest to use a construction from [HK12] which we summarize here. Consider a metric space  $(X, d)$ . A bi-infinite sequence  $\mathbf{P} = \{\mathbf{P}_n : n \in \mathbb{Z}\}$  of partitions of  $X$  is said to be a *hierarchical system* if  $\mathbf{P}_n$  is a refinement of  $\mathbf{P}_{n+1}$  for all  $n \in \mathbb{Z}$ . We say that  $\mathbf{P}$  is  $\Delta$ -adic if

$$S \in \mathbf{P}_n \implies \text{diam}_{(X,d)}(S) \leq \Delta^n \quad \forall n \in \mathbb{Z}.$$

**Theorem 1.13** ([HK12]). *Suppose  $(X, d)$  is a doubling metric space. Then there are numbers  $k, \ell, \Delta \geq 2$  that depend only on  $\mathcal{D}(X, d)$  such that the following holds. There is a collection  $\{\mathbf{P}^{(1)}, \dots, \mathbf{P}^{(k)}\}$  of  $\Delta$ -adic hierarchical systems such that for every subset  $S \subseteq X$  with  $\text{diam}_{(X,d)}(S) \leq \Delta^m$ , there is a set*

$$Q \in \bigcup_{i=1}^k \mathbf{P}_{m+\ell}^{(i)}$$

such that  $S \subseteq Q$ .

## 2 Discrete conformal metrics on $d$ -dimensional graphs

We first introduce a more general setting for graphs “packed” in  $d$ -dimensional spaces.

**QS packings.** Consider a (complete, separable) metric measure space  $(X, \text{dist}, \mu)$ . A Borel set  $S \subseteq X$  is said to be  $\gamma$ -round if the following holds: For every ball  $B$  in  $X$  whose center lies in  $\bar{S}$  (the closure of  $S$ ) and for which  $S \not\subseteq B$ , we have  $\mu(S \cap B) \geq \gamma \cdot \mu(B)$ . For instance, balls in  $\mathbb{R}^d$  are  $2^{-d}$ -round.

Consider additionally a collection  $\mathcal{F}_0$  of Borel sets in  $X$ . For  $\varepsilon > 0$ , and an at most countable subcollection  $\mathcal{F} \subseteq \mathcal{F}_0$  of pairwise-disjoint sets, define the  $\varepsilon$ -QS tangency graph  $G_\varepsilon[\mathcal{F}]$  as the graph with vertex set  $\mathcal{F}$  and with an edge between distinct sets  $S, T \in \mathcal{F}$  whenever

$$\text{dist}(S, T) \leq \varepsilon \cdot \min\{\text{diam}(S), \text{diam}(T)\}.$$

Say that a locally-finite graph  $G$  is  $\varepsilon$ -packable in  $(X, \text{dist}, \mu)$  by  $\mathcal{F}_0$  if  $G_\varepsilon[\mathcal{F}]$  is isomorphic to  $G$  for some subset  $\mathcal{F} \subseteq \mathcal{F}_0$ . Say that  $G$  is  $(\varepsilon, \gamma)$ -packable in  $(X, \text{dist}, \mu)$  if it is  $\varepsilon$ -packable by  $\gamma$ -round subsets of  $X$ . Say that  $G$  is QS-packable in  $(X, \text{dist}, \mu)$  if it is  $(\varepsilon, \gamma)$ -packable for some  $\varepsilon \geq 0, \gamma > 0$ . For instance, the tangency graph of a sphere-packing in  $\mathbb{R}^d$  is 0-packable by open Euclidean balls, and  $(0, 2^{-d})$ -packable in  $\mathbb{R}^d$ .

We now present the main theorem of this section. Despite the use of the symbol  $\varepsilon$ , one should observe it holds even for  $\varepsilon$  large. For the remainder of this section, we use the definition

$$d_* := \max(d, 2).$$

**Theorem 2.1.** *For every  $d \geq 1, c_1, c_2 > 0, \varepsilon \geq 0, \gamma > 0$ , there is a constant  $C$  such that the following holds. Suppose  $G = (V, E)$  is a finite graph that is  $(\varepsilon, \gamma)$ -packed in a  $(c_1, c_2, d)$ -regular space  $(X, \text{dist}, \mu)$ . Then for every  $R \geq 0$ , there is a conformal weight  $\omega : V \rightarrow \mathbb{R}_+$  that satisfies*

$$\frac{1}{|V|} \sum_{x \in V} \omega(x)^{d_*} = 1, \tag{2.1}$$

and such that

$$\max_{x \in V(G)} |B_\omega(x, R)| \leq CR^{d_*}. \tag{2.2}$$

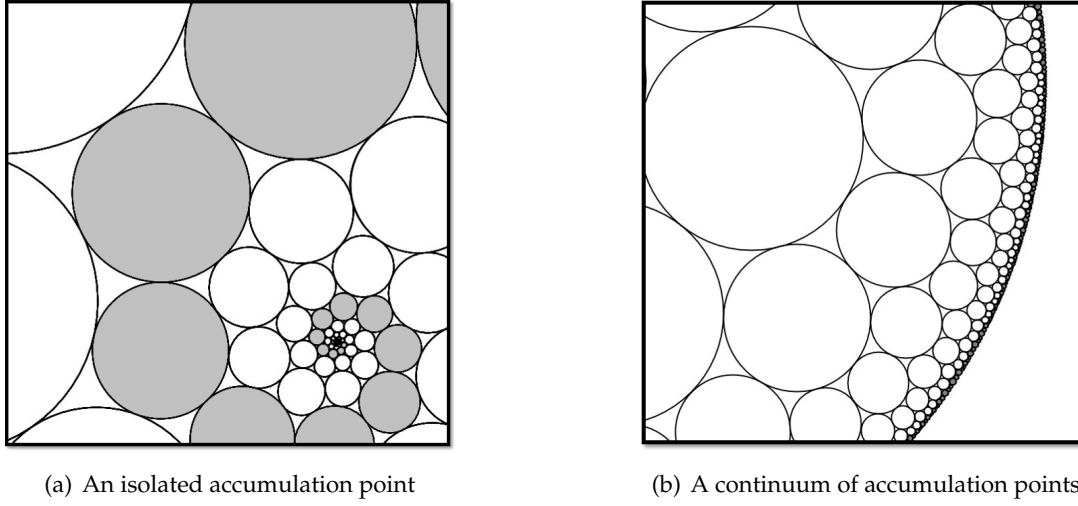


Figure 1: Accumulation points

Say that a sequence of graphs  $\{G_n\}$  is *uniformly QS-packable* in  $(X, \text{dist}, \mu)$  if there are numbers  $\varepsilon \geq 0, \gamma > 0$  such that  $G_n$  is  $(\varepsilon, \gamma)$ -packable in  $(X, \text{dist}, \mu)$  for every  $n \geq 1$ . Combining this with [Theorem 1.9](#) yields the following.

**Corollary 2.2.** *Consider an Ahlfors  $d$ -regular space  $(X, \text{dist}, \mu)$ . Suppose  $\{G_n\}$  is a sequence of finite graphs such that  $\{G_n\} \Rightarrow (G, \rho)$ . If  $\{G_n\}$  are uniformly QS-packable in  $(X, \text{dist}, \mu)$ , then  $(G, \rho)$  has gauged  $d_*$ -dimensional conformal growth.*

## 2.1 QS packings and the conformal weight

Our argument is inspired by the “isolation lemma” of Benjamini and Schramm [[BS01](#)] (see also [[BC11](#), [Gil14](#)]). Suppose  $G = (V, E)$  is sphere-packed in  $\mathbb{R}^d$ . When the spheres  $\{S_v : v \in V\}$  in the packing have similar radii, the background Euclidean metric provides a reasonable conformal weight; one sets  $\omega(v)$  proportional to the radius of the sphere  $S_v$ .

Difficulties arise when the radii degenerate, for instance near an accumulation point (in the case of infinite  $G$ ); see, for example, [Figure 1\(a\)](#). But if one imagines an *isolated* accumulation point as a cone, then it becomes rather tame: If we think of it as a metric on  $\mathbb{S}^{d-1} \times [0, \infty)$ , where the  $d$ th dimension is along the axis of the cone, then we merely need to do a “1-dimensional uniformization” along the axis (this can be seen in the use of the concavity of  $x \mapsto x^{1/d}$  in [Corollary 2.10](#) below). It would be problematic if the accumulation points themselves accumulated, e.g., as for a circle packing of a triangulation of the hyperbolic plane (e.g., [Figure 1\(b\)](#)). But the Benjamini-Schramm lemma asserts that this cannot happen for distributional limits of finite graphs packed in  $\mathbb{R}^d$ .

Since the result is trivial for  $d = 1$ , we assume that  $d > 1$ . Throughout the proof, we will use the asymptotic notation  $A \lesssim B$  to denote that  $A \leq C \cdot B$  for some constant  $C$  that depends only the parameters  $d, c_1, c_2, \varepsilon, \gamma$ . By default, we use the notation  $\text{diam}(\cdot)$  to denote the diameter in the metric  $\text{dist}$ . When we consider another metric, it will be explicitly specified.

We will need the following elementary fact which states that a point in an Ahlfors  $d$ -regular space cannot be near too many pairwise-disjoint  $\gamma$ -round bodies of large diameter.

**Lemma 2.3.** *If  $(X, \text{dist}, \mu)$  is  $(c_1, c_2, d)$ -regular, then for every  $\gamma > 0, \alpha \geq 1$ , and  $x \in X$ , the following holds. Suppose that  $S_1, S_2, \dots, S_K \subseteq X$  are pairwise-disjoint,  $\gamma$ -round sets satisfying*

$$\max_{i \in [K]} \text{dist}(x, S_i) < \alpha \cdot \min_{i \in [K]} \text{diam}(S_i),$$

*Then,*

$$K \leq \frac{c_2}{c_1 \gamma} (1 + 2\alpha)^d.$$

*Proof.* Let  $\lambda = \max_{i \in [K]} \text{dist}(x, S_i)$ , and let  $\{x_i\}$  be a collection of points such that  $x_i \in \bar{S}_i$  and  $\text{dist}(x, x_i) \leq \lambda$ . Consider the balls  $B_i = B(x_i, \lambda/2\alpha)$ . By assumption,  $\text{diam}(S_i) > \lambda/\alpha$ , hence  $S_i \not\subseteq B_i$ . Thus by the definition of  $\gamma$ -round,

$$\mu(S_i \cap B_i) \geq \gamma \mu(B_i) \geq \gamma c_1 (\lambda/2\alpha)^d,$$

where the latter inequality follows from the Ahlfors regularity of  $(X, \text{dist}, \mu)$ . But the sets  $\{S_i\}$  are pairwise-disjoint, and  $S_i \cap B_i \subseteq B(x, \lambda(1 + 1/2\alpha))$  for every  $i \in [K]$ , hence

$$K \gamma c_1 (\lambda/2\alpha)^d \leq \mu(B(x, \lambda(1 + 1/2\alpha))) \leq c_2 \lambda^d (1 + 1/2\alpha)^d,$$

where again the final inequality uses the Ahlfors  $d$ -regularity.  $\square$

### 2.1.1 Construction of the conformal weight

Suppose now that  $G = (V, E)$  is a finite graph that is  $(\varepsilon, \gamma)$ -packed in  $(X, \text{dist}, \mu)$ . Assume that  $k \geq 3$  is given. We will establish the existence of a metric  $\omega : V \rightarrow \mathbb{R}_+$  that satisfies  $\frac{1}{|V|} \sum_{x \in V} \omega(x)^d \lesssim 1$  and such that any subset  $U \subseteq V$  with  $|U| = 2^k$  satisfies  $\text{diam}_\omega(U) \gtrsim 2^{k/d_*}$ . This suffices to establish [Theorem 2.1](#).

For each vertex  $v \in V$ , there is an associated  $\gamma$ -round set  $S_v \subseteq X$ . Identify  $v$  with an arbitrary point in  $S_v$  so that we may consider  $V \subseteq X$ . Define  $\omega_0(v) = \mu(S_v)^{1/d}$ . For each  $S_v$ , consider a ball  $B$  of radius  $\text{diam}(S_v)/3$  whose center lies in  $S_v$  and such that  $S_v \not\subseteq B$ . Then since  $S_v$  is  $\gamma$ -round, we have

$$\text{diam}(S_v) \gtrsim \omega_0(v) = (\mu(S_v))^{1/d} \geq (\gamma \mu(B))^{1/d} \gtrsim \text{diam}(S_v), \quad (2.3)$$

where the first and last inequalities use Ahlfors  $d$ -regularity.

Let  $\mathbf{P} = \{\mathbf{P}_n : n \in \mathbb{Z}\}$  denote a  $\Delta$ -adic hierarchical system in  $X$  (recall [Section 1.7](#)). Define

$$\hat{\mathbf{P}} := \{(C, n) : n \in \mathbb{Z}, C \in \mathbf{P}_n\}.$$

Consider a positive integer  $s \lesssim 1$  to be chosen soon.

**The level of a cube.** For a pair  $(C, n) \in \hat{\mathbf{P}}$ , define

$$\text{lev}_{\mathbf{P}}(C, n) := \max \left\{ j \in \mathbb{N} : |(V \cap C) \setminus C'| \geq 2^j \text{ for all } C' \in \mathbf{P}_{n-s} \right\}.$$

The relevance of this definition is as follows. If  $\text{lev}_{\mathbf{P}}(C, n) = j$ , then we are witnessing a “feature” of size  $\approx 2^j$  that will not be fully seen by any cube at any lower scale. (For technical reasons, we actually shift by  $s$  scales, but  $s \lesssim 1$ .)

Thus we need to “uniformize” this feature at the current scale. Since we are trying to ensure  $d$ -dimensional volume growth, it should not be that this set of  $2^j$  points is contained in a set of  $\text{dist}_\omega$ -diameter significantly less than  $2^{j/d}$  (for  $d \geq 2$ ).

Let us first present a heuristic analysis. If we are in a cube  $C \in P_n$  of diameter at most  $\Delta^n$ , then we should rescale the metric by  $\approx \Delta^{-n} 2^{j/d}$  to ensure that we inflate this set to large enough diameter. (This is assuming that  $\text{diam}(V \cap C) \approx \Delta^n$ ; if the bulk of the set has much smaller diameter, this feature will be detected at the correct scale in some other hierarchical system.) Thus we should endow the vertices  $v \in V \cap C$  with weight  $\omega(v) \geq \beta \omega_0(v)$ , where  $\beta \approx \Delta^{-n} 2^{j/d}$ .

Consider now how much conformal weight we have spent. The total  $\ell_d$ -weight allocated is proportional to

$$\Delta^{-nd} 2^j \sum_{v \in V \cap C} \omega_0(v)^d \lesssim \Delta^{-nd} 2^j \Delta^{nd} \lesssim 2^j,$$

where we have employed (2.3), Ahlfors  $d$ -regularity, and the fact that the sets  $\{S_v\}$  are pairwise disjoint and contained in a ball of radius  $O(\Delta^n)$ .

Thus if we hope to keep the total  $\ell_d$ -weight bounded, it should be that we cannot detect too many level- $j$  features. This is the content of the next lemma which follows [BS01, Lem 2.3].

**Lemma 2.4.** *For all integers  $j \geq 0$ ,*

$$\#\{(C, n) \in \hat{P} : \text{lev}_P(C, n) = j\} \leq \frac{2s|V|}{2^j}. \quad (2.4)$$

*Proof.* Fix  $j \geq 0$ . Denote

$$[\sigma] := \{n \in \mathbb{Z} : n \equiv \sigma \pmod{s}\}.$$

We will prove that for  $\sigma \in \{0, 1, \dots, s-1\}$ ,

$$\#\{(C, n) \in \hat{P} : \text{lev}_P(C, n) = j \text{ and } n \in [\sigma]\} \leq \frac{2|V|}{2^j}. \quad (2.5)$$

Fix  $\sigma \in \{0, 1, \dots, s-1\}$ . For a pair  $(C, n) \in \hat{P}$ , define the set of children

$$\Lambda(C, n) := \{C' \subseteq C : C' \in P_{n-s}\}.$$

Define a “flow”  $F : (2^X \times [\sigma]) \times (2^X \times [\sigma])$  “up” the hierarchical system  $P$  as follows: For every  $n \in [\sigma]$ ,

$$F((C', n-s), (C, n)) = \begin{cases} \min\{2^{j-1}, |C' \cap V|\} & C \in P_n, C' \in \Lambda(C, n) \\ 0 & \text{otherwise.} \end{cases}$$

Define also:

$$\begin{aligned} F_{\text{in}}(C, n) &:= \sum_{(C', n') \in \hat{P}} F((C', n'), (C, n)), \\ F_{\text{out}}(C, n) &:= \sum_{(C', n') \in \hat{P}} F((C, n), (C', n')), \\ F_{\text{in}}^{(n)} &:= \sum_{C \in P_n} F_{\text{in}}(C, n). \end{aligned}$$

We make three observations:

1. First, notice that flow only goes “up” from a child set to a parent set, and thus from a lower level to a higher level:

$$F((C', n'), (C, n)) > 0 \implies n, n' \in [\sigma], n = n' + s, C' \in \Lambda(C, n).$$

2. The flow out of  $(C, n)$  is always at most the flow into  $(C, n)$ :  $F_{\text{out}}(C, n) \leq F_{\text{in}}(C, n)$ . This is because for  $C \in P_n$ ,

$$\sum_{C' \in \Lambda(C, n)} |C' \cap V| = |C \cap V|.$$

3. When  $\text{lev}_P(C, n) = j$ , the flow leaving  $(C, n)$  is less than the flow entering  $(C, n)$  by a least  $2^{j-1}$  because by definition of  $\text{lev}_P(C, n)$ ,

$$\sum_{C' \in \Lambda(C, n)} \min\{2^{j-1}, |C' \cap V|\} \geq 2^j.$$

In particular, combining this with observation (2) yields, for every  $n \in \mathbb{Z}$ ,

$$F_{\text{in}}^{(n-1)} \leq F_{\text{in}}^{(n)} - 2^{j-1} \#\{C \in P_n : \text{lev}_P(C, n) = j\}. \quad (2.6)$$

On the other hand, let  $n_0 \in [\sigma]$  be small enough so that every  $C \in P_{n_0}$  contains at most one point of  $V$ . Then  $F_{\text{in}}^{(n_0)} \geq |V|$ . Combining this with (2.6) and the fact that  $F \geq 0$  implies (2.5).  $\square$

Let us now assume additionally that  $P$  is  $\Delta$ -adic for some  $2 \leq \Delta \lesssim 1$  to be fixed momentarily. Given  $S \subseteq X$  and a parameter  $n \in \mathbb{Z}$ , we define the enlargements

$$\begin{aligned} N(S, R) &:= \{x \in X : \text{dist}(x, S) \leq R\} \\ \lambda(S, n) &:= N(S, 2\Delta^n). \end{aligned}$$

We define also the truncated level function:

$$\text{lev}_P^*(C, n) := \min\{\text{lev}_P(C, n), k\}.$$

Note that Lemma 2.4 gives

$$\#\{(C, n) \in \hat{P} : \text{lev}_P^*(C, n) = j\} \leq \frac{4s|V|}{2^j}, \quad (2.7)$$

where the extra factor of 2 comes from the consequence

$$\#\{(C, n) \in \hat{P} : \text{lev}_P(C, n) \geq j\} \leq \frac{4s|V|}{2^j}.$$

Recall that  $d_* = \max(d, 2)$ . For every  $(C, n) \in \hat{P}$ , we define a function  $\theta_P^{(C, n)} : V \rightarrow \mathbb{R}$  as follows:

$$\theta_P^{(C, n)}(v) = \begin{cases} \frac{2^{\text{lev}_P^*(C, n)/d_*}}{(1 + k - \text{lev}_P^*(C, n))^{2/d_*}} \cdot \min\left\{\Delta^{-n}, \frac{1}{\omega_0(v)}\right\} & \text{if } S_v \cap \lambda(C, n) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Define a conformal weight  $\omega_P : V \rightarrow \mathbb{R}_+$  by

$$\omega_P(v) := \omega_0(v) \left( \sum_{(C, n) \in \hat{P}} \left( \theta_P^{(C, n)}(v) \right)^{d_*} \right)^{1/d_*}$$

Denote

$$E_n(C) := \left\{ v \in V : \omega_0(v) > \Delta^n \text{ and } S_v \cap \lambda(C, n) \neq \emptyset \right\}, \quad (2.9)$$

From (2.3), we have  $v \in E_n(C) \implies \text{diam}(S_v) \gtrsim \omega_0(v) \gtrsim \Delta^n$ , and therefore Lemma 2.3 implies that

$$|E_n(C)| \lesssim 1 \quad \text{for all } (C, n) \in \hat{P}. \quad (2.10)$$

Now write:

$$\sum_{x \in V} \omega_P(v)^{d_*} = \sum_{j=0}^k \frac{2^j}{(1+k-j)^2} \sum_{n \in \mathbb{Z}} \sum_{\substack{C \in P_n: \\ \text{lev}_P^*(C, n) = j}} \left( |E_n(C)| + \Delta^{-d_* n} \sum_{v \notin E_n(C)} \omega_0(v)^{d_*} \right). \quad (2.11)$$

From (2.3), we have  $\text{diam}(S_v) \leq K_0 \omega_0(v)$  for some  $K_0 \lesssim 1$  and every  $v \in V$ . Thus in the case  $d = d_*$ , for a fixed  $C \in P_n$ , we have

$$\begin{aligned} \Delta^{-d_* n} \sum_{v \notin E_n(C)} \omega_0(v)^{d_*} &= \Delta^{-d_* n} \sum_{\substack{v \in V: \\ S_v \cap \lambda(C, n) \neq \emptyset \\ \omega_0(v) \leq \Delta^n}} \omega_0(v)^{d_*} \\ &= \Delta^{-dn} \sum_{\substack{v \in V: \\ S_v \cap \lambda(C, n) \neq \emptyset \\ \omega_0(v) \leq \Delta^n}} \mu(S_v) \\ &\lesssim \Delta^{-dn} \mu(\{x \in X : \text{dist}(x, C) \leq (2 + K_0) \Delta^n\}) \\ &\lesssim 1, \end{aligned}$$

where we have used Ahlfors  $d$ -regularity and the fact that the sets  $\{S_v\}$  are pairwise disjoint.

When  $d < d_*$ , one should write instead

$$\sum_{\substack{v \in V: \\ S_v \cap \lambda(C, n) \neq \emptyset \\ \omega_0(v) \leq \Delta^n}} \left( \frac{\omega_0(v)}{\Delta^n} \right)^{d_*} \lesssim \sum_{\substack{v \in V: \\ S_v \cap \lambda(C, n) \neq \emptyset \\ \omega_0(v) \leq \Delta^n}} \left( \frac{\omega_0(v)}{\Delta^n} \right)^d = \Delta^{-dn} \sum_{\substack{v \in V: \\ S_v \cap \lambda(C, n) \neq \emptyset \\ \omega_0(v) \leq \Delta^n}} \mu(S_v) \lesssim 1.$$

Using this in (2.11) together with (2.10), we conclude that

$$\begin{aligned} \sum_{x \in V} \omega_P(x)^{d_*} &\lesssim \sum_{j=0}^k \frac{2^j}{(1+k-j)^2} \# \{(C, n) \in \hat{P} : \text{lev}_P^*(C, n) = j\} \\ &\stackrel{(2.7)}{\leq} |V| \sum_{j=0}^k \frac{4s}{(1+k-j)^2} \\ &\lesssim |V|. \end{aligned} \quad (2.12)$$

Since  $(X, \text{dist})$  is doubling, Theorem 1.13 implies that for some  $k, \ell, \lesssim 1$  and  $2 \leq \Delta \lesssim 1$ , there is a sequence  $\{P^{(1)}, \dots, P^{(k)}\}$  of  $\Delta$ -adic hierarchical systems in  $X$  such that:

$$S \subseteq X, \text{diam}_{(X, d)}(S) \leq \Delta^m \implies S \subseteq C \text{ for some } (C, m + \ell) \in \bigcup_{i=1}^k \hat{P}^{(i)}. \quad (2.13)$$

Let us now set

$$s := \ell + 4$$

in the preceding construction. To construct our final weight, we define

$$\omega := \omega_{P^{(1)}} + \dots + \omega_{P^{(k)}}. \quad (2.14)$$



It follows that

$$\left( \frac{1}{|V|} \sum_{x \in V} \omega(x)^{d_*} \right)^{1/d_*} \lesssim \max \left\{ \left( \frac{1}{|V|} \sum_{x \in V} \omega_{P^{(i)}}(x)^{d_*} \right)^{1/d_*} : i = 1, \dots, k \right\} \stackrel{(2.12)}{\lesssim} 1,$$

where in the first inequality we used the fact that  $k \lesssim 1$ .

Thus the following lemma finishes the proof.

**Lemma 2.5.** *There is a number  $c \lesssim 1$  such that for every subset of vertices  $U \subseteq V$  with  $|U| = 2^k$ , there is an index  $i \in \{1, \dots, k\}$  satisfying*

$$\text{diam}_{\omega_{P^{(i)}}}(U) \gtrsim 2^{k/d_*}. \quad (2.15)$$

*Proof.* Let us fix a subset  $U \subseteq V$ , and denote  $D = \text{diam}(U) > 0$ . Let  $n' = \lceil \log_{\Delta} D \rceil + \ell$ . Then by (2.13), there is an index  $i \in \{1, \dots, k\}$  such that  $U \subseteq C$  for some  $(C, n') \in \hat{P}^{(i)}$ . Let  $P = P^{(i)}$ .

We now define inductively a sequence of pairs  $(C'_0, n'_0), (C'_1, n'_1), \dots, (C'_{m'}, n'_{m'}) \in \hat{P}$  as follows. Let  $\sigma = n'$  and recall that  $[\sigma] = \{n \in \mathbb{Z} : n \equiv \sigma \pmod{s}\}$ .

- Let  $C'_0 = C$  and  $n'_0 = \min\{n \in [\sigma] : (C'_0, n) \in \hat{P}\}$ .
- If  $|U \cap C'_i| \leq 1$ , we set  $m' = i$  and stop.

Otherwise, we choose  $C'_{i+1} \in P_{n_i-s}$  to be an element of the set  $\{C' \in P_{n_i-s} : C' \subseteq C'_i\}$  that maximizes  $|U \cap C'|$ . We define  $n'_{i+1} = \min\{n \in [\sigma] : (C'_{i+1}, n) \in \hat{P}\}$ .

Let us then pass to a maximal subsequence  $\{(C_0, n_0), (C_1, n_1), \dots, (C_m, n_m)\}$  of  $\{(C'_0, n'_0), \dots, (C'_{m'}, n'_{m'})\}$  with  $n_0 > n_1 > \dots > n_m$  and the property that

$$n_i = \min\{n \in [\sigma] : \exists (C'_j, n'_j) \text{ with } n = n'_j \text{ and } C'_j \cap U = C_i \cap U\}.$$

In other words, we enforce the property that

$$C_{i+1} \cap U \neq C_i \cap U \quad \text{for each } i = 0, 1, \dots, m-1. \quad (2.16)$$

Define  $C_{m+1} = \emptyset$ .

Note first that for every  $i \in \{0, 1, \dots, m\}$ ,

$$\text{lev}_P^*(C_i, n_i) \geq \lfloor \log_2 |(U \cap C_i) \setminus C_{i+1}| \rfloor. \quad (2.17)$$

From our choice of  $s = \ell + 4$  and the fact that  $P$  is  $\Delta$ -adic with  $\Delta \geq 2$ , it holds that

$$\text{diam}(C_1) \leq \Delta^{n'-s} \leq \Delta^{-3} D \leq \frac{D}{8}.$$

Since  $\text{diam}(U) = D$ , there must exist some  $u_0 \in U$  such that

$$\text{dist}(u_0, C_1) > \frac{D}{4} > \Delta^{n_1}. \quad (2.18)$$

Fix also some  $u_m \in C_m \cap U$ . We will establish that  $\text{dist}_{\omega_P}(u_1, u_m)$  is large, certifying that  $\text{diam}_{\omega_P}(U)$  is large as well.

Let  $N_i := |(U \cap C_i) \setminus C_{i+1}|$  for  $i = 0, 1, \dots, m$ . Note that  $N_i \geq 1$  from (2.16). Define

$$\ell_i := \text{lev}_P^*(C_i, n_i) \quad \text{for } i \in \{0, 1, \dots, m\},$$

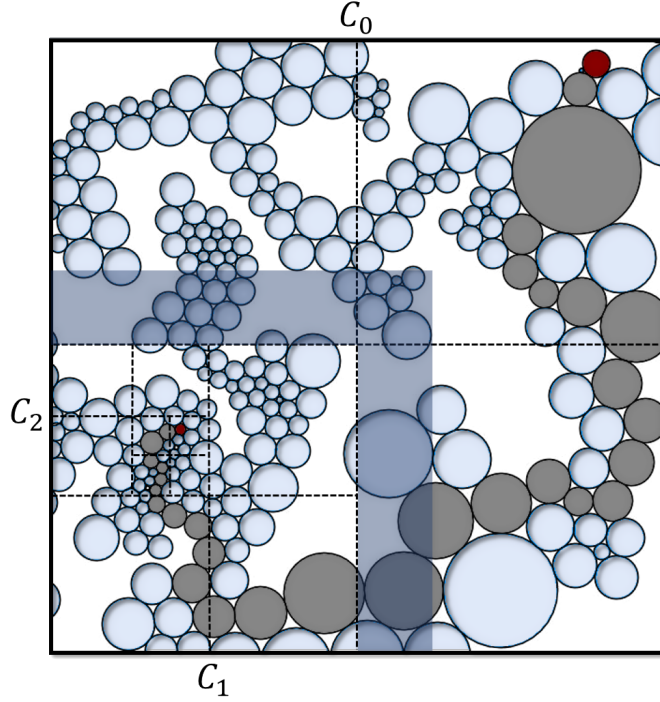


Figure 2: The path  $\gamma$  from  $u_0 \in C_0$  to  $u_m \in C_m$  passing through  $N(C_1, \Delta^{n_1}) \setminus C_1$ .

and observe from (2.17) that

$$2^{\ell_i} \geq N_i/2. \quad (2.19)$$

And by construction,

$$\sum_{i=0}^m N_i = |U| = 2^k. \quad (2.20)$$

**The length of a  $u_0$ - $u_m$  path.** Let  $\gamma = \langle v_0, v_1, v_2, \dots, v_t \rangle$  be an arbitrary simple path in  $G$  with  $v_0 = u_0$  and  $v_t = u_m$ . Our goal is to prove that

$$\text{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}, \quad (2.21)$$

since if this holds for all such paths  $\gamma$ , it verifies (2.15).

The basic outline is as follows. Informally, imagine that  $\gamma$  is parameterized by arclength in the metric  $\text{dist}$ . While  $\gamma$  need not spend much time in a cube  $C_i$ , it must cross from outside  $C_{i-1}$  to inside  $C_i$ , and therefore it must spend time  $\asymp \Delta^{n_i}$  in the neighborhood  $N(C_i, \Delta^{n_i})$ , where its  $\text{dist}_{\omega_P}$ -length experiences a reweighting by  $\theta_P^{(C_i, n_i)}$ . See Figure 2. We will now split  $\gamma$  into subpaths  $\gamma_0, \gamma_1, \dots, \gamma_m$  accordingly and show that the reweighting is sufficient to yield (2.21).

For  $i \in \{1, \dots, m\}$ , let  $s_i$  denote the largest index for which  $v_{s_i} \in \gamma$  satisfies  $v_{s_i} \notin N(C_i, \Delta^{n_i})$ , and let  $t_i$  denote the smallest index for which  $t_i > s_i$  and  $v_{t_i} \in N(C_i, \Delta^{n_i}/2)$ . Such indices must exist because  $\gamma$  begins at  $u_0 \notin N(C_1, \Delta^{n_1})$  (recall (2.18)) and  $\gamma$  ends at  $u_m \in C_m$ . Let  $\gamma_i$  denote the subpath  $\langle v_{s_i}, \dots, v_{t_i} \rangle$ . Define  $\gamma_0$  similarly unless  $\gamma \subseteq N(C_0, \Delta^{n_0})$ . In that case, we define  $\gamma_0 := \gamma$ . Observe that, by construction,

$$\text{len}_{\text{dist}}(\gamma_i) \gtrsim \Delta^{n_i}. \quad (2.22)$$

For  $i \geq 1$ , this follows from  $v_{s_i} \notin N(C_i, \Delta^{n_i})$  but  $v_{t_i} \in N(C_i, \Delta^{n_i}/2)$ . If  $i = 0$  and  $\gamma_0 = \gamma$ , it follows from

$$\text{len}_{\text{dist}}(\gamma) \geq \text{dist}(u_0, u_m) \stackrel{(2.18)}{\geq} D/4 \gtrsim \Delta^{n_0}.$$

This yields a lower bound on the  $\omega_0$ -length of each  $\gamma_i$ .

**Lemma 2.6.** *For each  $i \in \{0, 1, \dots, m\}$ ,*

$$\text{len}_{\omega_0}(\gamma_i) \gtrsim \Delta^{n_i}.$$

*Proof.* Parameterize  $\gamma_i = \langle x_1, x_2, \dots, x_h \rangle$ . Note that since  $G$  is  $(\varepsilon, \gamma)$ -packed, we have  $\text{dist}(S_{x_j}, S_{x_{j+1}}) \lesssim \text{diam}(S_{x_j})$ . Since  $x_j \in S_{x_j}, x_{j+1} \in S_{x_{j+1}}$ , it holds that

$$\begin{aligned} \text{dist}(x_j, x_{j+1}) &\leq \text{diam}(S_{x_j}) + \text{diam}(S_{x_{j+1}}) + \text{dist}(S_{x_j}, S_{x_{j+1}}) \\ &\lesssim \text{diam}(S_{x_j}) + \text{diam}(S_{x_{j+1}}) \\ &\lesssim \omega_0(x_j) + \omega_0(x_{j+1}), \end{aligned} \tag{2.23}$$

where the last inequality is (2.3).

We conclude that

$$\text{len}_{\omega_0}(\gamma_i) = \sum_{j=1}^h \omega_0(x_j) \stackrel{(2.23)}{\gtrsim} \text{len}_{\text{dist}}(\gamma_i) \gtrsim \Delta^{n_i}. \quad \square$$

Toward proving (2.21), observe that

$$\text{len}_{\omega_P}(\gamma) \geq \frac{1}{2} \sum_{j=0}^t \omega_P(v_j) \geq \frac{1}{2} \sum_{j=0}^t \omega_0(v_j) \left( \sum_{i=0}^m \left( \theta_P^{(C_i, n_i)}(v_j) \right)^{d_*} \right)^{1/d_*} \tag{2.24}$$

Let  $K_0 \lesssim 1$  be such that  $\text{diam}(S_v) \leq K_0 \omega_0(v)$  for all  $v \in V$ . Note that such a  $K_0$  exists from (2.3). For each  $v \in V$ , denote

$$L(v) := \left\{ i \in \{0, 1, \dots, m\} : \omega_0(v) > \frac{\Delta^{n_i}}{4\varepsilon K_0} \text{ and } S_v \cap \lambda(C_i, n_i) \neq \emptyset \right\}.$$

Define the subset

$$\Lambda := \left\{ i \in \{0, 1, \dots, m\} : i \notin \bigcup_{v \in V} L(v) \right\},$$

and the quantities

$$\begin{aligned} N_\Lambda &:= \sum_{i \in \Lambda} N_i \\ N_{\bar{\Lambda}} &:= 2^k - N_\Lambda. \end{aligned}$$

Clearly the following two claims suffice to establish (2.21).

**Lemma 2.7.** *If  $N_{\bar{\Lambda}} \geq 2^{k-1}$ , then*

$$\text{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}.$$

**Lemma 2.8.** *If  $N_\Lambda \geq 2^{k-1}$ , then*

$$\text{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}.$$

We will need the following elementary estimate.

**Lemma 2.9.** For some integer  $A \geq 2$ , consider  $S_A = \{(a_0, a_1, \dots, a_k) \in \mathbb{Z}_+^{k+1} : A = a_0 2^k + a_1 2^{k-1} + \dots + a_k\}$ . Then the quantity

$$\sum_{i=0}^k a_i \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}} \quad (2.25)$$

is minimized over  $S_A$  when  $a_1, \dots, a_k \in \{0, 1\}$ .

*Proof.* Consider any  $(a_0, a_1, \dots, a_k) \in S_A$  such that  $a_i \geq 2$  for some  $i > 0$ . Then  $(a'_0, a'_1, \dots, a'_k) \in S_A$  where  $a'_j = a_j$  if  $j \notin \{i, i-1\}$ , and  $a'_i = a_i - 2$ ,  $a'_{i-1} = a_{i-1} + 1$ . We can calculate the change in the value of (2.25):

$$\begin{aligned} \frac{2^{(k-i)/d_*}}{(6+i)^{2/d_*}} - 2 \frac{2^{(k-(i+1))/d_*}}{(7+i)^{2/d_*}} &= 2^{(k-i)/d_*} \left( \frac{1}{(6+i)^{2/d_*}} - \frac{2^{1-1/d_*}}{(7+i)^{2/d_*}} \right) \\ &= \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}} \left( \left(1 + \frac{1}{6+i}\right)^{2/d_*} - 2^{1-1/d_*} \right) < 0, \end{aligned}$$

where we have used  $d_* \geq 2$ . □

**Corollary 2.10.** Suppose that for some  $a_0, a_1, a_2, \dots, a_k \in \mathbb{Z}_+$ , it holds that  $a_0 2^k + a_1 2^{k-1} + \dots + a_k \geq 2^{k-2}$ . Then,

$$\sum_{i=0}^k a_i \frac{2^{(k-i)/d_*}}{(1+i)^{2/d_*}} \geq \frac{2^{(k-2)/d_*}}{9}.$$

*Proof.* Applying Lemma 2.9 gives

$$\sum_{i=0}^k a_i \frac{2^{(k-i)/d_*}}{(1+i)^{2/d_*}} \geq \sum_{i=0}^k a_i \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}} \geq \frac{2^{(k-2)/d_*}}{9^{2/d_*}} \geq \frac{2^{(k-2)/d_*}}{9}. \quad \square$$

*Proof of Lemma 2.7.* From the definition (2.8), we have

$$i \in L(v) \implies \left( \omega_0(v) \theta_P^{(C_i, n_i)}(v) \right)^{d_*} \gtrsim \frac{2^{\ell_i}}{(1+k-\ell_i)^2} \quad (2.26)$$

Using (2.24) in conjunction with (2.26) yields

$$\text{len}_{\omega_P}(\gamma) \gtrsim \sum_{v \in \gamma} \left( \sum_{i \in L(v)} \frac{2^{\ell_i}}{(1+k-\ell_i)^2} \right)^{1/d_*} \geq \sum_{v \in \gamma} \sum_{i \in L(v)} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d_*}}. \quad (2.27)$$

Now from (2.19), we have

$$\sum_{v \in \gamma} \sum_{i \in L(v)} 2^{\ell_i} \geq N_{\bar{\Lambda}}/2 \geq 2^{k-2}.$$

Thus Corollary 2.10 in conjunction with (2.27) yields the desired bound. □

*Proof of Lemma 2.8.* Let  $\Gamma_j := \{i \in \Lambda \setminus \{0\} : v_j \in \gamma_i\}$ . Then using Hölder's inequality in (2.24) yields

$$\begin{aligned} \text{len}_{\omega_P}(\gamma) &\geq \frac{1}{2} \sum_{j=0}^t \omega_0(v_j) \left( \sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \left( \theta_P^{(C_i, n_i)}(v_j) \right)^{d_*} \right)^{1/d_*} \\ &\geq \sum_{j=0}^t \omega_0(v_j) (1 + |\Gamma_j|)^{-1/d_*} \sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \theta_P^{(C_i, n_i)}(v_j). \end{aligned} \quad (2.28)$$

**Lemma 2.11.** For every  $i \in \Lambda$ , if  $\gamma_i = \langle x_1, \dots, x_h \rangle$ , then

$$\text{dist}(x_j, x_{j+1}) \leq \frac{\Delta^{n_i}}{4} \quad \text{for } j = 1, 2, \dots, h-1.$$

In particular, it holds that  $\gamma_i \subseteq N(C_i, 2\Delta^{n_i}) \setminus N(C_i, \Delta^{n_i}/4)$ .

*Proof.* By construction, we have  $x_2, \dots, x_h \in N(C_i, \Delta^{n_i})$  and  $x_1, \dots, x_{h-1} \notin N(C_i, \Delta^{n_i}/2)$ . Thus the second assertion of the lemma follows from the first.

To verify the former, note that since  $x_2, \dots, x_h \in N(C_i, \Delta^{n_i})$ , we have  $S_{x_j} \cap \lambda(C_i, n_i) \neq \emptyset$  for  $j = 2, \dots, h$ , and therefore since  $i \in \Lambda$ , we have  $\omega_0(x_j) < \frac{\Delta^{n_i}}{4\varepsilon K_0}$ . The  $(\varepsilon, \gamma)$ -packing property guarantees that, for  $j = 0, 1, \dots, h-1$ , since  $\{x_j, x_{j+1}\} \in E(G)$ ,

$$\text{dist}(x_j, x_{j+1}) \leq \varepsilon \text{diam}(S_{x_{j+1}}) \leq \varepsilon K_0 \omega_0(v_{j+1}) \leq \varepsilon K_0 \frac{\Delta^{n_i}}{4\varepsilon K_0} \leq \frac{\Delta^{n_i}}{4}. \quad \square$$

**Lemma 2.12.** For each  $j \in \{0, 1, \dots, t\}$ , it holds that  $|\Gamma_j| \leq 1$ .

*Proof.* Note that since  $n_{i+1} \leq n_i - s$  for all  $i = 0, 1, \dots, m-1$ , and  $\Delta \geq 2, s \geq 4$ , the sets  $N(C_i, 2\Delta^{n_i}) \setminus N(C_i, \Delta^{n_i}/4)$  are pairwise disjoint for all  $i = 0, 1, \dots, m$ . Hence the result follows from [Lemma 2.11](#).  $\square$

Using this in (2.28) gives

$$\begin{aligned} \text{len}_{\omega_P}(\gamma) &\geq \sum_{j=0}^t \omega_0(v_j) \sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \theta_P^{(C_i, n_i)}(v_j) \\ &= \sum_{i \in \Lambda} \sum_{v \in \gamma_i} \theta_P^{(C_i, n_i)}(v) \omega_0(v). \end{aligned} \quad (2.29)$$

From [Lemma 2.6](#), we know that

$$\sum_{v \in \gamma_i} \omega_0(v) \geq \Delta^{n_i}. \quad (2.30)$$

For  $i \in \Lambda$ , [Lemma 2.11](#) yields  $\gamma_i \subseteq N(C_i, 2\Delta^{n_i}) = \lambda(C_i, n_i)$ , hence  $S_v \cap \lambda(C_i, n_i) \neq \emptyset$  for each  $v \in \gamma_i$ . From the definition of  $\Lambda$ , this yields  $\omega_0(v) \leq \frac{\Delta^{n_i}}{4\varepsilon K_0}$ , thus from the definition (2.8),

$$v \in \gamma_i \implies \theta_P^{(C_i, n_i)}(v) \geq \Delta^{-n_i} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d_*}}.$$

Combining this with (2.29) and (2.30) gives

$$\text{len}_{\omega_P}(\gamma) \geq \sum_{i \in \Lambda} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d_*}}.$$

By (2.19) and our assumption that  $N_\Lambda = \sum_{i \in \Lambda} N_i \geq 2^{k-1}$ , we have  $\sum_{i \in \Lambda} 2^{\ell_i} \geq 2^{k-2}$ . Thus [Corollary 2.10](#) gives

$$\text{len}_{\omega_P}(\gamma) \geq 2^{k/d_*},$$

completing the proof.  $\square$

$\square$

## 2.2 $d$ -parabolicity

We first discuss two examples showing that for distributional limits of finite graphs with uniformly bounded degrees,  $d$ -parabolicity and the property that  $\overline{\dim}_{\text{cg}}^d(G, \rho) \leq d$  are incomparable.

**Example 2.13.** Let  $\{H_n\}$  denote an infinite family of transitive, 3-regular expander graphs with  $|V(H_n)| \in [n, 2n]$ . Let  $H$  denote a random finite graph such that  $\mathbb{P}[H = H_n] = c/n^2$  for all  $n \geq 1$  and the appropriately chosen constant  $c > 0$ . Consider now a bi-infinite path  $P$  and identify each  $v \in V(P)$  with a vertex in an independent copy  $H_v$  of  $H$ . Let  $G$  denote the resulting random graph. Fix a node  $v_0 \in V(P)$  and choose a root  $\rho \in H_{v_0}$  uniformly at random. The random rooted graph  $(G, \rho)$  is unimodular, and the weighting  $\omega(v) = \frac{1_{V(P)}(v)}{1 + \text{dist}_G(v_0, v)}$  demonstrates that  $G$  is almost surely 2-parabolic. On the other hand, it is not difficult to verify that  $\underline{\dim}_{\text{cg}}(G, \rho) = \infty$ .

**Example 2.14.** For the other direction, we construct an example for  $d = 2$ . Consider a random tree  $T_n$  of height  $n$  rooted at  $v_0 \in V(T_n)$ .  $T_n$  is described by the following branching process: A node at distance  $k$  from the root produces  $N_k \in \{1, 2\}$  offspring where

$$\mathbb{P}[N_k = 2] = \frac{1}{k} + \frac{1}{k\sqrt{\log k}}.$$

In particular, for any  $r \leq n$ , it holds that

$$\mathbb{E} \left[ |B_{T_n}(v_0, r)| \right] \asymp r^2 e^{\sqrt{\log r}},$$

and a simple tail bound yields

$$\mathbb{E} \left[ \sup_{r \leq n} |B_{T_n}(v_0, r)| \right] \asymp r^2 e^{\sqrt{\log r}}.$$

Let  $(T, \rho)$  denote the distributional limit of  $(T_n, \rho_n)$  where  $\rho_n \in V(T_n)$  is chosen uniformly at random. The uniform metric  $\omega \equiv 1$  exhibits  $\overline{\dim}_{\text{cg}}(G, \rho) \leq 2$ .

On the other hand, almost surely the optimal unit flow from  $\rho_n$  to infinity has energy at most

$$O(1) \sum_{r \geq 1} \frac{1}{r e^{\sqrt{\log r}}} < \infty,$$

implying that  $T$  is almost surely transient. Since 2-parabolicity is equivalent to recurrence for bounded degree graphs, this shows that  $T$  is almost surely not 2-parabolic.

One can construct similar examples for  $d > 2$ . The connection to transience is not necessary; one simply uses the duality formula between  $L^d$  extremal length and the infimal “ $L^{d/(d-1)}$  energy” of a unit flow from  $v_0$  to  $\infty$ .

### 2.2.1 Gauged conformal growth and vertex extremal length

We now prove that gauged  $d$ -dimensional conformal growth implies  $d$ -parabolicity.

*Proof of Theorem 1.5.* Fix  $d \geq 2$  and a unimodular random graph  $(G, \rho)$  with gauged  $d$ -dimensional conformal growth and such that  $\deg_G(\rho)$  is essentially bounded. For each  $R \geq 0$ , let  $\omega_R$  be an  $L^d$ -normalized conformal metric on  $(G, \rho)$  that satisfies

$$\|B_{\omega_R}(\rho, R)\|_{L^\infty} \leq CR^d \tag{2.31}$$

for some constant  $C \geq 1$ .

From [Lee17a, Lem. 2.6], we may assume that for each  $R \geq 0$ , the following additional properties hold almost surely:

1. For all  $x \in V(G)$ ,  $\omega_R(x) \geq 1/2$ .
2. For all  $\{x, y\} \in E(G)$ , we have  $\omega_R(x) \leq C' \omega_R(y)$ , where  $C'$  is a constant depending only on  $\|\deg_G(\rho)\|_{L^\infty}$ .

Moreover, these additional properties are sufficient to guarantee that we can compare  $\text{dist}_{\omega_R}$  balls to  $\text{dist}_G$  balls in the following sense (see [Lee17a, Lem. 2.5]): Almost surely, for every  $x \in V(G)$  and  $R, r \geq 0$ ,

$$B_G\left(x, \frac{\log \frac{r}{2\omega_R(x)}}{\log C'}\right) \subseteq B_{\omega_R}(x, r) \subseteq B_G(x, 2r). \quad (2.32)$$

Fix  $\varepsilon \in (0, 1)$ ,  $n \geq 1$ . Let  $\{r_j\}$  be the sequence of numbers  $r_1 = 1$  and for  $j > 1$ ,

$$\frac{\log \frac{\varepsilon r_j}{16C'}}{\log C'} = 2r_{j-1}.$$

Denote

$$\Lambda_G := \left\{x \in V(G) : \omega_{r_j}(x) \leq \frac{1}{\varepsilon} \text{ for } j \leq n\right\}.$$

For  $x \in V(G)$ , let

$$A_j(x) := B_{\omega_{r_j}}(x, r_j) \setminus B_{\omega_{r_j}}\left(x, \frac{r_j}{8C'}\right).$$

By our choice of the sequence  $\{r_j\}$  and (2.32), for every  $x \in \Lambda_G$ , we have

$$B_{\omega_{r_{j-1}}}(x, r_{j-1}) \subseteq B_G(x, 2r_{j-1}^{j-1}) \subseteq B_{\omega_{r_j}}(x, r_j/(8C')), \quad (2.33)$$

hence if  $x \in \Lambda_G$ , then the sets  $A_1(x), A_2(x), \dots, A_n(x)$  are pairwise disjoint.

Consider now the following conformal weight (which depends on the root  $\rho$ ):

$$\omega_\rho(x) := \left( \sum_{j=1}^n r_j^{-d} \omega_{r_j}(x)^d \mathbf{1}_{A_j(\rho)}(x) \right)^{1/d}.$$

By construction,

$$\mathbb{E} \left[ \sum_{x \in V(G)} \omega_\rho(x)^d \mid \rho \in \Lambda_G \right] \leq \sum_{j=0}^n r_j^{-d} \mathbb{E} \left[ \mathcal{V}_{\omega_{r_j}}(\rho, r_j) \mid \rho \in \Lambda_G \right], \quad (2.34)$$

where

$$\mathcal{V}_\omega(x, r) := \sum_{y \in B_\omega(x, r)} \omega(y)^d,$$

and we used the fact that  $\rho \in \Lambda_G$  implies that the sets  $A_j(\rho)$  are pairwise disjoint for  $j = 1, 2, \dots, n$  from (2.33).

Now observe that

$$\text{dist}_{\omega_\rho}(\rho, x) \geq \sum_{j=1}^n \frac{\text{dist}_{\omega_{r_j}} \mathbf{1}_{A_j(\rho)}(\rho, x)}{r_j}.$$



Suppose that  $x \in V(G) \setminus B_G(\rho, 2r_n)$  and consider any path  $\gamma$  from  $\rho$  to  $x$  in  $G$ . Let  $\hat{\gamma}$  denote the portion which lies inside  $A_j(\rho)$ . Every vertex  $u \in B_{\omega_j}(\rho, r_j/(8C'))$  satisfies  $\omega_{r_j}(u) \leq r_j/(4C')$  by definition of  $\text{dist}_{\omega_{r_j}}$ , thus if  $\{u, v\} \in E(G)$ , then by Property (2) above,  $\omega(v) \leq r_j/4$ .

In particular,

$$\text{len}_{\omega_{r_j} \mathbf{1}_{A_j(\rho)}}(\gamma) = \text{len}_{\omega_{r_j}}(\hat{\gamma}) \geq \frac{r_j}{2C'}.$$

We conclude that

$$\rho \in \Lambda_G \implies \text{dist}_{\omega_\rho}(\rho, V(G) \setminus B_G(\rho, 2r_n)) \geq \sum_{j=1}^n \frac{r_j}{2C'r_j} \geq \frac{n}{2C'}. \quad (2.35)$$

Let us now return to (2.34). For a conformal metric  $\omega : V(G) \rightarrow \mathbb{R}_+$  and some  $R > 0$ , define the transport

$$F(G, \omega, x, y) = \omega(x)^d \mathbf{1}_{\{\text{dist}_\omega(x, y) \leq R\}}.$$

Then by the Mass-Transport Principle,

$$\begin{aligned} \mathbb{E}[\mathcal{V}_\omega(x, R)] &= \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, x, \rho)\right] = \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, \rho, x)\right] \\ &= \mathbb{E}\left[\omega(\rho)^d |B_\omega(\rho, R)|\right] \leq \|B_\omega(\rho, R)\|_{L^\infty} \mathbb{E}\left[\omega(\rho)^d\right]. \end{aligned}$$

We conclude from (2.31) that for each  $j \leq n$ ,

$$\mathbb{E}\left[\mathcal{V}_{\omega_{r_j}}(\rho, r_j) \mid \rho \in \Lambda_G\right] \leq \frac{Cr_j^d}{\mathbb{P}[\rho \in \Lambda_G]},$$

hence

$$\mathbb{E}\left[\sum_{x \in V(G)} \omega_\rho(x)^d \mid \rho \in \Lambda_G\right] \leq \frac{Cn}{1 - \varepsilon^d n},$$

where we have used Markov's inequality and a union bound to assert that  $\mathbb{P}[\rho \in \Lambda_G] \geq 1 - \varepsilon^d n$ .

Take  $\varepsilon = 1/n$  and  $n \geq 2$  in the preceding construction and define the event

$$\mathcal{E}(n) := \left\{ \omega_{r_j}(\rho) \leq n \text{ for } j \leq n \text{ and } \|\omega_\rho\|_{\ell_d^d(V(G))}^d \leq 2Cn^{1.5} \right\}.$$

By Markov's inequality and a union bound, we have

$$\mathbb{P}(\mathcal{E}(n)) \geq \frac{2}{\sqrt{n}}.$$

Moreover from (2.35),

$$\mathcal{E}(n) \implies \frac{\text{dist}_{\omega_\rho}(\rho, V(G) \setminus B_G(\rho, 2r_n))}{\|\omega_\rho\|_{\ell_d^d(V(G))}} \geq \frac{n}{4C'C^{1/d}n^{1.5/d}} \geq \frac{n^{1/4}}{4C'\sqrt{C}}.$$

In other words, for every  $n \geq 1$ , it holds that

$$\mathbb{P}\left[\text{VEL}_d(\Gamma_G(\rho)) \geq \frac{n^{1/4}}{4C'\sqrt{C}}\right] \geq 1 - \frac{2}{\sqrt{n}}.$$

It follows that

$$\mathbb{P}[\text{VEL}_d(\Gamma_G(\rho)) = \infty] = 1,$$

i.e., almost surely  $G$  is  $d$ -parabolic.  $\square$

### 2.3 Spectral bounds for the graph Laplacian

We now prove the following generalization of [Theorem 1.7](#).

**Theorem 2.15.** *For every  $d \geq 1$ ,  $c_1, c_2 > 0$ ,  $\varepsilon \geq 0$ ,  $\gamma > 0$ , there is a constant  $c$  such that the following holds. Suppose  $G = (V, E)$  is a finite graph that is  $(\varepsilon, \gamma)$ -packed in a  $(c_1, c_2, d)$ -regular space  $(X, \text{dist}, \mu)$ . If  $G$  is an  $n$ -vertex graph that can be sphere-packed in  $\mathbb{R}^d$ , then for  $k = 1, 2, \dots, n-1$ ,*

$$\lambda_k(G) \leq c \frac{\Delta_G(k)}{k} \left( \log \frac{n}{k} \right)^2 \left( \frac{k}{n} \right)^{2/d}.$$

Consider a finite connected graph  $G = (V, E)$ . Define the Rayleigh quotient  $\mathcal{R}_G(f)$  of non-zero  $f : V \rightarrow \mathbb{R}$  by

$$\mathcal{R}_G(f) := \frac{\sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{\sum_{x \in V} \deg_G(x) f(x)^2}.$$

It is an elementary fact (see, e.g., [[Lee17a](#), Cor. 3.1]) that to establish [Theorem 2.15](#), it suffices to find  $k$  disjointly supported functions  $\varphi_1, \varphi_2, \dots, \varphi_k : V \rightarrow \mathbb{R}$  such that for each  $i = 1, 2, \dots, k$ ,

$$\mathcal{R}_G(\varphi_i) \leq c \frac{\Delta_G(k)}{k} \left( \log \frac{n}{k} \right)^2 \left( \frac{k}{n} \right)^{2/d}.$$

Toward this end, we now state [[Lee17a](#), Thm. 3.13]. For a finite graph  $G = (V, E)$ , denote  $\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon|V|)}{\varepsilon|V|}$ .

**Theorem 2.16.** *There is a constant  $C \geq 1$ . Consider a finite graph  $G = (V, E)$  with  $n = |V|$ . Suppose that  $\omega : V \rightarrow \mathbb{R}_+$  is a conformal metric on  $G$  satisfying,*

1.  $\frac{1}{|V|} \sum_{x \in V} \omega(x)^2 \leq 1$ .
2. For some numbers  $R > 0, K \geq 2$ :

$$\max_{x \in V} |B_\omega(x, R)| \leq K \leq n/2. \quad (2.36)$$

Then there exist disjoint supported functions  $\varphi_1, \varphi_2, \dots, \varphi_k : V \rightarrow \mathbb{R}_+$  with  $k \geq n/16K$ , and such that

$$\max \{\mathcal{R}_G(\varphi_1), \dots, \mathcal{R}_G(\varphi_k)\} \leq C \frac{(\log K)^2 (\bar{d}_G(1/K) + \bar{d}_G(1/R^2))}{R^2}.$$

Now [Theorem 2.15](#) is a consequence of the following proposition combined with [Theorem 2.1](#).

**Proposition 2.17.** *Suppose that  $G = (V, E)$  is an  $n$ -vertex graph with  $(c, R, d)$ -growth for some numbers  $c \geq 1, d \geq 2$  and all  $R \geq 0$ . Then for  $k = 1, 2, \dots, n-1$ ,*

$$\lambda_k(G) \leq O(1) \frac{\Delta_G(k)}{k} \left( \log \frac{n}{k} \right)^2 \left( \frac{ck}{n} \right)^{2/d}.$$

*Proof.* For each  $R \geq 0$ , let  $\omega_R : V \rightarrow \mathbb{R}_+$  be a conformal metric on  $G$  satisfying

$$\frac{1}{|V|} \sum_{x \in V} \omega_R(x)^d = 1,$$

and

$$\max_{x \in V} |B_\omega(x, R)| \leq cR^d.$$

Note that from Hölder's inequality,

$$\frac{1}{|V|} \sum_{x \in V} \omega_R(x)^2 \leq \left( \frac{1}{|V|} \sum_{x \in V} \omega_R(x)^d \right)^{2/d} = 1.$$

So we can apply [Theorem 2.16](#) with  $\omega_R$  and  $K = cR^d$  to obtain, for  $k \leq n/(16cR^d)$ ,

$$\lambda_k(G) \leq O(1) \frac{(d \log R)^2 \bar{d}_G(\frac{1}{cR^d})}{R^2}.$$

Setting  $R := (n/16ck)^{1/d}$  yields

$$\lambda_k(G) \leq O(1) \left( \frac{ck}{n} \right)^{2/d} \left( \log \frac{n}{k} \right)^2 \frac{\Delta_G(k)}{k}.$$

completing the proof. □

### 3 The systolic minor problem

One might try to replace the geometric approach of the preceding section with one based on intrinsic considerations, i.e., properties of a graph that do not refer directly to an ambient geometric representation. [Conjecture 3.1](#) below would provide an alternate proof of our main theorem (in the special case of sphere-packings in  $\mathbb{R}^d$ ) in conjunction with the flow-crossing theory (see [\[Lee17b\]](#)).

We refer to this as the “systolic minor” problem for the following reason. Given a Riemannian manifold  $(M, g)$ , a non-contractible curve is a witness to the fact that  $M$  is not simply-connected. Systolic geometry (see, e.g., the recent survey [\[Gut10\]](#)) asks about the shortest length of such a witnesses.

In this light, consider the fact that a graph is planar if and only if it excludes  $K_{3,3}$  and  $K_5$  as minors. One might ask for the smallest “length” of a witness, and such a notion has been studied under the name of *shallow minors* [\[PRS94\]](#). A graph  $G$  contains an  $H$  minor if and only if there are pairwise-disjoint, connected subsets  $\{A_v \subseteq V(G) : v \in V(H)\}$  such that  $\{u, v\} \in E(H) \implies E_G(A_u, A_v) \neq \emptyset$ , i.e. there is at least one edge between the sets  $A_u$  and  $A_v$ . Say that a graph  $G$  contains an  $H$  minor at depth  $D$  if additionally one can choose the sets  $\{A_v\}$  so that  $\text{diam}_G(A_v) \leq D$  for every  $v \in V(H)$ .

Fix  $d \geq 3$ . It is straightforward to see that, for every finite graph  $H$ , there is a finite graph  $G$  that contains  $H$  as a minor and such that  $G$  can be sphere-packed in  $\mathbb{R}^d$ . On the other hand, it was shown by Lachlan in 1886 that if  $G$  can be sphere packed in  $\mathbb{R}^d$ , then it cannot contain a  $K_{d+3}$  minor<sup>1</sup> at depth 0. In other words, at most  $d + 2$  interior-disjoint spheres can be mutually tangent in  $\mathbb{R}^d$ . The next conjecture asserts a lower bound on the length of complete graph minors in  $\mathbb{R}^d$ .

**Conjecture 3.1.** *Let  $G$  be the tangency graph of a sphere packing in  $\mathbb{R}^d$  for  $d \geq 3$ . For  $h \geq d + 3$ , suppose that  $\{A_j : j = 1, 2, \dots, h\}$  is a collection of pairwise-disjoint, connected subsets of  $G$  with the property that  $E_G(A_i, A_j) \neq \emptyset$  for all  $i \neq j$ . Then,*

$$\max \left\{ \text{diam}_G(A_j) : j = 1, 2, \dots, h \right\} \geq c_d \frac{h^{2/(d-2)}}{(\log h)^{O(1)}},$$

for some constant  $c_d$  depending on the dimension.

---

<sup>1</sup> $K_h$  denotes the complete graph on  $h$  vertices.

**Remark 3.2.** Using fairly simple volume arguments, one can prove this with the right-hand side replaced by  $h^{1/2(d+1)}$  [PRS94]. It is open for the case  $d = 3$ , even if one allows  $G$  to be packed using all spheres of the same radius.

**Remark 3.3** (A tight example). In  $\mathbb{R}^3$ , it is possible to sphere-pack a graph  $G$  that has a  $K_h$  minor at depth  $O(h^2(\log h)^2)$ , and thus the conjectured bound is nearly tight. This can be achieved using all spheres of the same radius. We sketch the construction.

First, observe that any  $n$ -vertex, 4-regular graph  $G$  has a subdivision that can be sphere packed in  $\mathbb{R}^3$  with unit spheres and such that every edge is subdivided into a path of length  $O(n)$ . To do this, place the vertices of the graph in the  $z = 0$  plane equally spaced on a circle of radius  $n$ . Decompose the edges of  $G$  into two perfect matchings; use positive  $z$ -coordinates for the edges in one matching and negative  $z$ -coordinates for the edges in the other.

Consider one of the two matchings  $X \subseteq E(G)$  and enumerate  $X = \{e_1, \dots, e_{n/2}\}$ . Now “draw” an edge  $e_j = \{x_j, y_j\}$  by going straight up from  $x_j$  to height  $j$ , then along a straight line in the  $z = j$  plane to the point lying over  $y_j$ , then straight down to  $y_j$ . It is straightforward to see that one can fill in the “lines” with unit spheres so that each edge uses only  $O(n)$  spheres.

Now we are left to find a 4-regular graph  $G$  containing a small depth  $K_h$  minor for  $h$  large. It is known that a random 4-regular graph on  $n$  vertices contains a  $K_h$  minor with  $h \geq c \sqrt{n}$  for some constant  $c > 0$  [FKO09]. In general, if  $G$  is an expander graph (as will be the case for a random 4-regular graph with high probability), then it has a  $K_h$  minor at depth  $O(\log n)$  for  $h \geq c \sqrt{\frac{n}{\log n}}$  [PRS94]. Plugging this into the preceding construction yields an  $\mathbb{R}^3$ -packable graph  $G$  with a  $K_h$  minor at depth  $O(h^2(\log h)^2)$ .

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